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Projective and affine connections on S^1 and integrable systems

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Dedicated to Professor Nigel Hitchin on his 55th birthday

Abstract

It is known that the Korteweg–de Vries (KdV) equation is a geodesic flow of an L^2 metric on the Bott–Virasoro group. This can also be interpreted as a flow on the space of projective connections on S^1. The space of differential operators $\Delta^{(n)} = \partial^n + u_2\partial^{n-2} + \cdots + u_n$ form the space of extended or generalized projective connections. If a projective connection is factorizable $\Delta^{(n)} = (\partial - ((n+1)/2 - 1)p_1)\cdots(\partial + (n-1)/2p_n)$ with respect to quasi primary fields p_i's, then these fields satisfy $\sum_{i=1}^{n}((n+1)/2 - i)p_i = 0$. In this paper we discuss the factorization of projective connection in terms of affine connections. It is shown that the Burgers equation and derivative non-linear Schrödinger (DNLS) equation or the Kaup–Newell equation is the Euler–Arnold flow on the space of affine connections.

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1. Introduction

It is well known that the periodic KdV equation is the archetypal example of the Euler flow on the coadjoint orbit of the Bott–Virasoro group. This can be interpreted as a geodesic flow of the right invariant metric on the Bott–Virasoro group, which at the identity is given by the L^2-inner product [15,17,19].

It is well known that the KdV equation is the canonical example of a scalar Lax equation, which is an equation defined by a Lax pair of scalar differential operators

$$\frac{d\Delta^{(n)}}{dt} = [P, \Delta^{(n)}],$$

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where

$$\Delta^{(n)} = \frac{d^n}{dx^n} + u_{n-2}\frac{d^{n-1}}{dx^{n-2}} + \cdots + u_0.$$

Here P is a differential operator whose coefficients are differential polynomial in the variables, essentially determined by the requirement that $[P, L]$ be an operator of order less than n. The space of differential operators $\Delta^{(n)}$ is known as Adler–Gelfand–Dickey space or AGD space.

The action of $\mathrm{Diff}(S^1)$ on AGD space has been known since last century. Wilczynski [18] gave a description of the transformation of scalar differential operators by a change of variable of the independent variable x. Let $x \to \sigma(x)$ be a smooth change of variables. The action of $\mathrm{Diff}(S^1)$ on the AGD manifold associated to $SL(n, \mathbf{R})$ is given by

$$\kappa : \mathrm{Diff}(S^1) \times \mathrm{AGD} \to \mathrm{AGD}, \qquad (\sigma, \Delta^{(n)}) \mapsto \kappa(\sigma)\Delta^{(n)},$$

where $\kappa(\sigma)\Delta^{(n)}$ is the unique scalar differential operator of the form $u_{n-1} = 0$.

It is known that the operators $\Delta^{(n)}$ do not preserve their form under the action of $\mathrm{Diff}(S^1)$, $x \to \sigma(x)$, due to the appearance of the $(n-1)$th term $-1/2n(n-1)(\sigma''/\sigma'^{n+1})$. Hence we should think the operators are acting on *densities of weight* $-1/2(n-1)$ rather than on scalar functions, in this case we can always find $u_{n-1} = 0$ as a reparametrization invariant. Therefore, the action of $\mathrm{Diff}(S^1)$ on Δ^n is given by

$$\begin{aligned}&\partial_x^n + u_{n-2}(x)\partial_x^{n-2} + \cdots + u_0(x)\\ &\quad \to \sigma'^{-(n+1)/2}(\partial_x^n + \tilde{u}_{n-2}\partial_x^{n-2} + \cdots + \tilde{u}_0)\sigma'^{-(n-1)/2},\end{aligned} \tag{1}$$

where

$$\tilde{u}_{n-2} = \sigma'^2 u_{n-2}(\sigma(x)) + \tfrac{1}{12}n(n-1)(n+1)S'(x), \tag{2}$$

and $S(x, \sigma) = (\sigma'''/\sigma' - 3\sigma''^2/2\sigma'^2)$ is the Schwarzian of x and σ.

The action of $\mathrm{Diff}(S^1)$ transform the solutions of $\Delta^{(n)}\xi = 0$ as densities of degree $(n-1)/2$. If μ and ξ be the solutions of $\kappa(\sigma)\Delta^{(n)}$ and $\Delta^{(n)}$, then their solutions are related by

$$\mu = (\sigma'^{(n-1)/2}\xi) \circ \sigma.$$

In the case for $n = 2$, this coincides with the action of Virasoro group on the space of Hill operators, dual space of Virasoro algebra.

Let us define the Dickey's notion [3,4] of *quasi primary* field. If

$$u_{n-2} = \frac{\tilde{u}_{n-2}}{c_n}, \qquad c_n = \frac{1}{12}n(n-1)(n+1), \tag{3}$$

then we say u_{n-2} is a quasi primary field with conformal weight 2.

A quasi primary field of the weight 1 is a field $p(x)$ with a transformation law under diffeomorphism $x \mapsto \sigma(x)$

$$p(t) = p(x)\sigma + \chi, \qquad \chi = \frac{\sigma'}{\sigma}. \tag{4}$$

A quasi primary field of weight 2 is factorizable (we assume it) to quasi primary field of weight 1. Later we will see that this is related to Miura transformation.

Leonid Dickey showed that a quasi primary field u of the weight 2 can be represented as

$$u = -\tfrac{1}{2}p^2 + p',$$

where $p(x)$ is a quasi primary field of the weight 1. This quasi primary field will play a big role in our paper.

1.1. Goal and plan

Earlier we have studied the action of vector field $\mathrm{Vect}(S^1)$ on the space of Adler–Gelfand–Dickey operators [6–8]. This space is identified with the space of projective connections, initiated by Cartan [2]. It is known that all these operators are factorizable

$$\Delta^{(n)} = (\partial - v_1)\cdots(\partial - v_n),$$

where v_i's are Miura variables and satisfy $\sum_{i=1}^{n} v_i = 0$. It was shown by Dickey that all these Miura variables are connected to quasi primary fields.

In this paper we will consider the Burgers and derivative non-linear Schrödinger (DNLS) equations as an Euler–Arnold flows on the space of first-order linear differential operators. There are various versions of non-linear derivative Schrödinger equations, in this paper we will consider only the Kaup–Newell version of DNLS equation.

The product of two linear differential operators form a projective connection. Thus we say that the space of linear differential operators form a space of affine connections. When an operator or projective connection on circle is factorizable, the diffeomorphism group acts on it through affine connections. In this way we connect the flows on the space of projective connections and the affine connections.

The paper is arranged as follows. In Section 2 we review some background materials, such as the definition of projective connection, KdV equation as the Euler–Arnold flow [1] on the space of projective connections etc. In Section 3 we discuss affine connections and its $\mathrm{Vect}(S^1)$ module structure. We show that the Burgers and the Kaup–Newell equations as the Euler–Arnold type flows on the space of affine connections. The relation the KdV flows and the Burgers flows is given in Section 4.

In this paper we will also establish a natural relationship between the second Hamiltonian (or Poisson) operator of the KdV equation and the operator[1] of the Burgers equation. We show that the Poisson operator of the KdV equation is factorizable into the Poisson operator of the Burgers equation and some linear operator.

Let us state our result of the paper.

Theorem 1. *Let u be a quasi primary field of weight 2, given by $u = -1/2p^2 + p_x$, where p is a quasi primary field of weight 1. This induces the factorization of the projective connection* (*or Hill's operator*) $\Delta^{(2)} = \Delta^1\Delta_1$ *in terms of affine connections* (Δ^1 *or* Δ_1) *acting*

[1] This is a non-skew symmetric operator.

on the space of tensor densities of degree $\pm 1/2$, *given by*

$$\mathcal{F}_{1/2} \stackrel{\Delta_1}{\rightarrow} \mathcal{F}_{-1/2} \stackrel{\Delta^1}{\rightarrow} \mathcal{F}_{-3/2}.$$

(A) *This action of* $\mathrm{Vect}(S^1)$ *on the space of affine connections generates a Hamiltonian flow of Euler–Arnold type, and give rise to Burgers and Kaup–Newell equations.*

(B) *Suppose* $\mathcal{O}_{KdV} = 1/2\,\mathrm{d}^3/\mathrm{d}x^3 + 2u\,\mathrm{d}/\mathrm{d}x + u'$ *and* $\mathcal{O}_B = 1/2(\mathrm{d}^2/\mathrm{d}x^2 + v\,\mathrm{d}/\mathrm{d}x + v'$ *be the Hamiltonian structures for KdV and Burgers equation, respectively. Then the factorization* $\Delta^{(2)} = \Delta^1\Delta_1$ *induces a factorization*:

$$\mathcal{O}_{\mathrm{KdV}} = (\partial - v)\mathcal{O}_B,$$

for all $u = 1/2(v' - 1/2v^2)$.

2. Preliminaries

Let $\mathrm{Diff}(S^1)$ be the group of orientation preserving diffeomorphisms of the circle. It is known that the group $\mathrm{Diff}(S^1)$ as well as its Lie algebra of vector fields on S^1, $T_{id}\mathrm{Diff}(S^1) = \mathrm{Vect}(S^1)$, have non-trivial one-dimensional central extensions, the Bott–Virasoro group $\widehat{\mathrm{Diff}}(S^1)$ and the Virasoro algebra *Vir*, respectively [11,12].

The Lie algebra $\mathrm{Vect}(S^1)$ is the algebra of smooth vector fields on S^1. This satisfies the commutation relations

$$\left[f\frac{\mathrm{d}}{\mathrm{d}x}, g\frac{\mathrm{d}}{\mathrm{d}x}\right] := (f(x)g'(x) - f'(x)g(x))\frac{\mathrm{d}}{\mathrm{d}x}. \tag{5}$$

One parameter family of $\mathrm{Vect}(S^1)$ acts on the space of smooth functions $C^\infty(S^1)$ by

$$\mathcal{L}^{(\mu)}_{f(x)\,\mathrm{d}/\mathrm{d}x}a(x) = f(x)a'(x) - \mu f'(x)a(x), \tag{6}$$

where

$$\mathcal{L}^{(\mu)}_{f(x)\,\mathrm{d}/\mathrm{d}x} = f(x)\frac{\mathrm{d}}{\mathrm{d}x} - \mu f'(x) \tag{7}$$

is the derivative with respect to the vector field $f(x)\,\mathrm{d}/\mathrm{d}x$. Eq. (6) implies a one parameter family of $\mathrm{Vect}(S^1)$ action on the space of smooth functions $C^\infty(S^1)$.

Let us denote $\mathcal{F}_\mu(M)$ the space of tensor densities of degree $-\mu$

$$\mathcal{F}_\lambda = \{a(x)\,\mathrm{d}x^{-\lambda} | a(x) \in C^\infty(S^1)\}.$$

Thus, we say

$$\mathcal{F}_{-\lambda} \in \Gamma(\Omega^{\otimes\lambda}), \qquad \Omega^{\otimes\lambda} = (T^*S^1)^{\otimes\lambda},$$

where $\mathcal{F}_0(M) = C^\infty(M)$, the space $\mathcal{F}_{-1}(M)$ coincides with the space differential forms.

Hence, the equation can be interpreted as an action of $\mathrm{Vect}(S^1)$ on $\mathcal{F}_\mu(S^1)$, a tensor densities on S^1 of degree μ [14,16].

Thus Eq. (5) can be interpreted as an action of $\mathrm{Vect}(S^1)$ on $\mathcal{F}_1 \in \Gamma(\Omega^{-1})$. In this paper we will mainly consider the action of $\mathrm{Vect}(S^1)$ on $\mathcal{F}_{1/2} \in \Gamma(\Omega^{-1/2})$, square root of the tangent bundle

$$\mathcal{L}_{f(x)\,\mathrm{d}/\mathrm{d}x} h(x) = (f(x)h'(x) - \tfrac{1}{2} f'(x)h(x)), \tag{8}$$

where $h(x)\sqrt{\mathrm{d}/\mathrm{d}x} \in \Gamma(\Omega^{-1/2})$ [5,9].

2.1. Projective connection on the circle and KdV equation

Let us denote $\Omega^{\pm 1/2}$ be the square root of the cotangent and tangent bundle, respectively.

Definition 1. An extended projective connection on the circle is a class of differential (conformal) operators

$$\Delta^{(n)} : \Gamma(\Omega^{-(n-1)/2}) \to \Gamma(\Omega^{(n+1)/2}),$$

such that

1. The symbol of $\Delta^{(n)}$ is the identity,
2. $$\int_{S^1} (\Delta^{(n)} s_1) s_2 = \int_{S^1} s_1 (\Delta^{(n)} s_2)$$
   for all $s_i \in \Gamma(\Omega^{-(n-1)/2})$.

Hence the projective connection $\Delta^{(2)}$ can be identified with the Hill operator $\mathrm{d}^2/\mathrm{d}x^2 + u(x)$.

KdV equation. The space $C^\infty(S^1) \oplus \mathbf{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the *regular part*, and the pairing between this space and the Virasoro algebra is given by:

$$\left\langle (u(x), a), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, \alpha \right) \right\rangle = \int_{S^1} u(x) f(x)\,\mathrm{d}x + a\alpha.$$

It is well known that the Virasoro algebra is the unique (upto isomorphism) non-trivial central extension of $\mathrm{Vect}(S^1)$. It is given by the Gelfand–Fuchs cocycle [12]

$$\omega \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x} \right) = \int_{S^1} f'(x) g''(x)\,\mathrm{d}x.$$

The Virasoro algebra is therefore a Lie algebra on the space $\mathrm{Vect}(S^1) \oplus \mathbf{R}$.

$$\left[\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a \right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b \right) \right] = \left(\left[f\frac{\mathrm{d}}{\mathrm{d}x}, g\frac{\mathrm{d}}{\mathrm{d}x} \right] \mathrm{Vect}(S^1), \omega(f, g) \right). \tag{9}$$

The *regular part* of the dual of the Virasoro algebra is $C^\infty \oplus \mathbf{R}$, and a pairing between this space and Virasoro algebra is given by

$$\left\langle (u(x), c), \left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a \right) \right\rangle := \int_{S^1} u(x) f(x)\,\mathrm{d}x + ca.$$

It is almost trivial to find the KdV equation on the coadjoint orbit from this recipe.

By Lazutkin and Pankratova [13], this dual space can be identified with the space of Hill's operator or the space of projective connections (see [7,9] for details)

$$\Delta = \frac{d^2}{dx^2} + u(x), \tag{10}$$

where u is a periodic potential: $u(x + 2\pi) = u(x) \in C^\infty(\mathbb{R})$. The Hill's operator maps

$$\Delta : \mathcal{F}_{1/2} \to \mathcal{F}_{-3/2}. \tag{11}$$

The action of $\mathrm{Vect}(S^1)$ on the space of Hill's operator Δ is defined by the commutation with the Lie derivative

$$[\mathcal{L}_{f(x)\,d/dx}, \Delta] := \mathcal{L}^{-3/2}_{f(x)\,d/dx} \circ \Delta - \Delta \circ \mathcal{L}^{1/2}. \tag{12}$$

Certainly, Eq. (12) is the coadjoint action of $\mathrm{Vect}(S^1)$. Hence, we can extract the operator $ad^*_{u(t)}$ from this information. The Euler–Arnold equation is the Hamiltonian flow on the coadjoint orbit on the space of Hill's operator, generated by the Hamiltonian

$$H(u) = \tfrac{1}{2}\langle u(x), u(x)\rangle, \tag{13}$$

given by

$$\frac{du}{dt} = -ad^*_{u(x,t)} u(x, t). \tag{14}$$

It must be noted that $ad^*_{u(x,t)}$ can be realized as the variational derivative of H. Hence it belongs to the space of Virasoro algebra, and it is given by

$$ad^*_{u(x,t)} = \frac{1}{2}\frac{d^3}{dx^3} + 2u\frac{d}{dx} + u'.$$

The KdV equation follows from this definition.

Leonid Dickey[2] took a different approach to study the action of vector field on the Adler–Gelfand–Dickey space. Let us relate our work with Dickey's work. Our formula (12) can be easily identified with Dickey's formula [3, Chapter 3]

$$V(P) = Q\Delta - \Delta P,$$

where

$$Q = (LPL^{-1})_+ = -\Delta P - (\Delta P \Delta^{-1})_+ \Delta = -(\Delta P \Delta^{-1})_-,$$

where P is identified with $\mathcal{L}_{f(x)\,d/dx}$.

The Euler–Arnold equation in this form is given by

$$\frac{d\Delta}{dt} = -(\Delta P \Delta^{-1})_-.$$

[2] Thanks to Professor Dickey for pointing us this connection.

3. Affine connection and Euler–Arnold flows

Definition 2. An affine connection on the circle is a linear first-order differential operator

$$\mathcal{D} : \Gamma(\Omega^{-1/2}) \to \Gamma(\Omega^{1/2}),$$

such that

1. The (formal) square of the $\mathcal{D}$ is the projective connection.
2. $(\mathcal{D}s_1)s_2 \in C^\infty(S^1)$, for all $s_i \in \Gamma(\Omega^{-1/2})$.

Since $\mathcal{D}$ maps one space to another, so square of $\mathcal{D}$ is not a precise statement. Hence later we will present a clear explanation.

Lemma-Definition 1. A first-order differential operator maps

$$\partial - \mu p(x) : \Gamma(\Omega^{\mu}) \to \Gamma(\Omega^{\mu+1}).$$

Proof. Let $g \in \mathcal{F}_\mu$ and $g_1(x) = (\partial - \mu p(x))$. Let us consider a transformation $x \to x(t)$

$$\begin{aligned} g_1(t) &= (\partial_t - \mu p(t)) = (\phi\partial_x - \mu(p(x)\phi + \chi)g(x)\phi^{\mu} \\ &= [(\partial - \mu p(x))g]\phi^{\mu+1} = g_1(x)\phi^{\mu+1}, \end{aligned}$$

where we have used $1/\phi\, \mathrm{d}/\mathrm{d}t = \mathrm{d}/\mathrm{d}x$. Hence $g_1(x) = (\partial - \mu p(x)) \in \Gamma(\Omega^{\mu+1})$. □

Let us denote $\mathcal{D}_\mu : \Gamma(\Omega^{\mu}) \to \Gamma(\Omega^{\mu+1})$.

Lemma 1.

$$\mathcal{D}_{\mu+\nu}(g_1 g_2) = (\mathcal{D}_{\mu g_1})g_2 + g_1(\mathcal{D}_{\nu g_2}),$$

where $g_1 \in \Gamma(\Omega^{\mu})$ *and* $g_2 \in \Gamma(\Omega^{\nu})$.

Proof. Let $g_1 \in \mathcal{F}_\mu$ and $g_2 \in \mathcal{F}_\nu$, then $g_1 g_2 \in \mathcal{F}_{\mu+\nu}$.

$$\begin{aligned} \mathcal{D}_{\mu+\nu}(g_1 g_2) &= (\partial - (\mu+\nu)p)(g_1 g_2) = (\partial - \mu p)g_1 \cdot g_2 + g_1 \cdot (\partial - \nu p)g_2 \\ &= (\mathcal{D}_{\mu g_1}) \cdot g_2 + g_1 \cdot (\mathcal{D}_{\nu g_2}). \end{aligned}$$

□

It is clear from the definition

$$\mathcal{D}_\mu^2 \equiv (\partial - (\mu+1)p)(\partial - \mu p) : \Gamma(\Omega^{\mu}) \to \Gamma(\Omega^{\mu+2}). \tag{15}$$

Remark 1. Hence we can identify $\mathcal{D}_\mu^2$ with Δ when $\mu = -1/2$.

Let us recall our convention. We denote $\mathcal{F}_{-(n-1)/2} \in \Gamma(\Omega^{(n-1)/2})$. Let us consider the factorization

$$\Delta^{(n)} : \mathcal{F}_{(n-1)/2} \to \mathcal{F}_{-(n+1)/2},$$
$$\mathcal{F}_{(n-1)/2} \overset{\left(\partial + \frac{n-1}{2} p_n\right)}{\to} \mathcal{F}_{(n+3)/2} \overset{\left(\partial + \frac{n-3}{2} p_{n-1}\right)}{\to} \cdots \overset{\left(\partial - \frac{n-1}{2}\right)}{\to} \mathcal{F}_{-(n+1)/2}, \tag{16}$$

where $p_1, \ldots, p_n$ are quasi primary fields, and the Miura variables are given by

$$v_i := \left(\frac{n+1}{2} - i\right) p_i. \tag{17}$$

Hence we say

$$\mathcal{D}_{(n-1)/2} \cdots \mathcal{D}_{-(n-1)/2} = \Delta^{(n)} : \mathcal{F}_{(n-1)/2} \to \mathcal{F}_{-(n+1)/2}. \tag{18}$$

3.1. *Flows on affine connections*

In this section we show that the Burgers flow and the Kaup–Newell flows are the Euler–Arnold flows on the space of affine connections.

Burgers flow: Let us consider an operator

$$\Delta_1 = \frac{d}{dx} + \frac{1}{2} A(x), \tag{19}$$

acting on $\mathcal{F}_{1/2} \in \Gamma(\Omega^{-1/2})$, square root of the tangent bundle on S^1.

This Δ_1 satisfies

$$\Delta_1 = \frac{d}{dx} + \frac{1}{2} A(z) : \mathcal{F}_{1/2} \to \mathcal{F}_{-1/2}. \tag{20}$$

Definition 3. The Vect(S^1)-action on Δ_1 is defined by the commutator with the Lie derivative

$$[\mathcal{L}_{f(x)\,d/dx}, \Delta_1] := \mathcal{L}^{1/2}_{f(x)\,d/dx} \circ \Delta_1 - \Delta_1 \circ \mathcal{L}^{-1/2}_{f(x)\,d/dx}. \tag{21}$$

The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

Lemma 2.

$$[\mathcal{L}_{f(x)\,d/dx}, \Delta_1] = f''(z) + Af'(x) + A' f(x). \tag{22}$$

Proof. By direct computation. □

Hence the operator is

$$\mathcal{O}_B = \frac{d^2}{dx^2} + A \frac{d}{dx} + A'(x). \tag{23}$$

Remark. Strictly speaking this operator is not a Poisson operator, since it does not satisfy the skew symmetric condition. When a vector field Vect(S^1) acts on the space of projective connections it generates a Poisson flow, and the operator involves in this flow is Poisson operator. But when Vect(S^1) acts on the space of affine connections it generates a Poisson flow, but the operator involved here is non-skew symmetric.

Hence we immediately get the Burgers equation from the Hamiltonian equation

$$A_t = \mathcal{O}_B \frac{\delta H}{\delta A}, \qquad H[A] = \frac{1}{2}A^2. \tag{24}$$

Thus we have the following lemma.

Lemma 3. *The Euler–Arnold flow on the space of first-order differential operator gives the Burgers equation*

$$A_t = 2AA_x + A_{xx}. \tag{25}$$

Kaup–Newell flow: Let us apply the above scheme in the holomorphic setting. Let us consider an operator

$$\Delta_1 = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z} - \frac{1}{2}|q|^2(z), \tag{26}$$

acting on vector valued functions $\psi : \mathbb{R} \to \mathbf{C}$, where for some smooth function $q : \mathbb{R} \to \mathbf{C}$ with compact support.

Let us consider the $\mathrm{Vect}_{\mathbf{C}}(S^1)$-action on Δ_1. The result of this action is a scalar operator, i.e. the operator of multiplication by a function.

Lemma 4.

$$[\mathcal{L}_{f(z)\,\mathrm{d}/\mathrm{d}z}, \Delta_1] = \tfrac{1}{2}(\mathrm{i}f''(z) - |q|^2 f'(z) - |q|_z^2 f(z)). \tag{27}$$

Proof. By direct computation. □

Hence the operator is

$$\mathcal{O}_{\mathrm{KN}} = \mathrm{i}\frac{\mathrm{d}^2}{\mathrm{d}z^2} - |q|^2\frac{\mathrm{d}}{\mathrm{d}z} - |q|_z^2. \tag{28}$$

If the Hamiltonian functional is given by

$$H = \int |q|^2\,\mathrm{d}z,$$

then $\delta H/\delta\bar{q} = q$; here we write $\delta/\delta q$ is the Euler–Lagrange operator: in general if f is a function of $(q, q_z, q_{zz}, \dots)$ then

$$\frac{\delta f}{\delta q} = \sum_0^\infty (-\partial)^i \frac{\partial f}{\partial q^{(i)}}.$$

Thus the Euler–Arnold flow is

$$q_t = \mathcal{O}_{\mathrm{KN}} \frac{\delta H}{\delta \bar{q}},$$

satisfies

$$q_{zz} + 2\mathrm{i}(|q|^2 q)_z + \mathrm{i}q_t = 0. \tag{29}$$

It is a well known DNLS equation, and this has been studied by Kaup–Newell [10]. Thus we prove the first part of our theorem.

4. Relation with the KdV flow

Let us consider Eq. (16). Let us choose

$$p_1 = p_2 = \cdots = p_n = p(x).$$

This leads to

$$\begin{aligned}\Delta^n &= \left(\partial - \left(\frac{n+1}{2} - 1\right) p(x)\right)\left(\partial - \left(\frac{n+1}{2} - 2\right) p(x)\right)\cdots\left(\partial + \frac{n-1}{2} p(x)\right)\\ &= \partial^n + c_n\left(p_x - \frac{p^2}{2}\right)\partial^{n-2} + \cdots\end{aligned}$$

Let us recall

$$\partial + \tfrac{1}{2}v : \mathcal{F}_{1/2} \to \mathcal{F}_{-1/2}, \qquad \partial - \tfrac{1}{2}v : \mathcal{F}_{-1/2} \to \mathcal{F}_{-3/2}.$$

Let us consider the Burgers equation. Let us recall the Poisson operator of Burgers equation

$$\mathcal{O}_B = \frac{1}{2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v\frac{\mathrm{d}}{\mathrm{d}x} + v'\right).$$

The Hill's operator is equivalent to the relation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{1}{2}v(x)\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}v(x)\right), \tag{30}$$

where

$$u = \tfrac{1}{2}(v_x - \tfrac{1}{2}v^2),$$

giving the formal factorization of the Hill's operator.

Geometrically this can be realized as

$$\mathcal{F}_{1/2} \xrightarrow{\Delta_1} \mathcal{F}_{-1/2} \xrightarrow{\Delta^1} \mathcal{F}_{-3/2},$$

where $\Delta = \Delta^1\Delta_1 = (\partial - 1/2v)(\partial + 1/2v)$. This is compatible with $\Delta : \mathcal{F}_{1/2} \to \mathcal{F}_{-3/2}$.

Proposition 1. *The Poisson operator of KdV equation is a factorizable into a Poisson operator of the Burgers flow and a first-order linear operator*.

Proof. This can be verified from

$$[\mathcal{L}_{f(x)\,\mathrm{d}/\mathrm{d}x}, \Delta] \equiv [\mathcal{L}_{f(x)\,\mathrm{d}/\mathrm{d}x}, \Delta^1\Delta_1] = (\partial - v)[\mathcal{L}_{f(x)\,\mathrm{d}/\mathrm{d}x}, \Delta_1] = (\partial - v)\tfrac{1}{2}(\partial^2 + v\partial + v') = (\partial - v)\tfrac{1}{2}\partial(\partial + v).$$

Thus we find that

$$(\partial - v)\frac{1}{2}\partial(\partial + v) \equiv \frac{1}{2}\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 2\left(v' - \frac{1}{2}v^2\right)\frac{\mathrm{d}}{\mathrm{d}x} + (v'' - vv')\right),$$

is compatible with the KdV's Poisson operator $1/2\,\mathrm{d}^3/\mathrm{d}x^3 + 2u\,\mathrm{d}/\mathrm{d}x + u'$, for $u = 1/2(v' - 1/2v^2)$.

Hence we say that a typical skew symmetric differential operator defining the second Hamiltonian structure of KdV is a product of the Hamiltonian structure of the Burgers operator and $(\partial - v)$, i.e.,

$$\mathcal{O}_{\mathrm{KdV}} = (\partial - v)\mathcal{O}_B.$$ □

Thus we establish a natural relationship between the KdV equation and the Kaup–Newell equation.

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References

[1] V.I. Arnold, Mathematical methods of classical mechanics, in: Graduate Texts in Mathematics, 2nd Edition, Vol. 60, Springer, Berlin, 1989.
[2] E. Cartan, Lecons sur la Theorie des Espaces a Connection Projective, Gauthier-Villars, Paris, 1937.
[3] L.A. Dickey, Lectures on classical W-algebras, Acta Appl. Math. 47 (1997) 243–321.
[4] L.A. Dickey, Soliton equations and Hamiltonian systems, in: Advanced Series in Mathematical Physics, Vol. 12, World Scientific, River Edge, NJ, 1991, x + 310 pp.
[5] P. Guha, Diffeomorphism, periodic KdV and C. Neumann system, Diff. Geom. Appl. 12 (2000) 1–8.
[6] P. Guha, Diff(S^1), Adler–Gelfand–Dickey spaces and integrable systems, Lett. Math. Phys. 55 (2001) 17–31.
[7] P. Guha, Diffeomorphism on S^1, projective structures and integrable systems, in: Proceedings of the Kruskal 2000 Conference, in press. Also the longer version is available (IHES/M/00/24). http://www.ihes.fr/information/preprintM00.html.
[8] P. Guha, Geometry of Kaup–Newell equation, Rep. Math. Phys., in press.

[9] N. Hitchin, Vector fields on the circle, in: M. Francaviglia (Ed.), Mechanics, Analysis and Geometry: 200 Years After Lagrange, Elsevier, Amsterdam, 1991.
[10] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19, 798–801.
[11] A.A. Kirillov, Infinite dimensional Lie groups: their orbits, invariants and representations, in: The Geometry of Moments, Lect. Notes in Math. 970, Springer, Berlin, 1982, pp. 101–123.
[12] A. Kirillov, The orbit method: I and II. Infinite-dimensional Lie groups and Lie algebras, Contemp. Math. 145 (1993).
[13] V.F. Lazutkin, T.F. Pankratova, Normal forms and the versal deformations for Hill's equation, Funct. Anal. Appl. 9 (4) (1975) 41–48.
[14] P. Marcel, V. Ovsienko, C. Roger, Extension of the Virasoro and Neveu–Schwartz algebras and generalized Sturm-Liouvilleoperators, Lett. Math. Phys. 40 (1997) 31–39.
[15] V. Ovsienko, B. Khesin, KdV super equation as an Euler equation, Funct. Anal. Appl. 21 (1987) 329–331.
[16] V. Ovsienko, C. Roger, Extensions of Virasoro group and Virasoro algebra by modules of tensor densities on S^1.
[17] G. Segal, Unitary representations of some infinite dimensional groups, Commun. Math. Phys. 80 (1981) 301–342.
[18] E.J. Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, Chelsea, New York, 1905.
[19] E. Witten, Coadjoint orbits of the Virasoro group, Commun. Math. Phys. 114 (1988) 1–53.